SOLUTION BY W. J. THOME, University of Detroit.

It is a well known fact that if the parallel sides of a trapezoid are g and s, and the perpendicular distance between them is p, then the distance from the greater side g to the center of gravity of the trapezoid is

$$\frac{p}{3}\,\frac{(g+2s)}{(g+s)}.$$

Let t be the constant width of the lever, in inches;
d, the density of the material of the lever,
M, the total mass of the lever;
m_1, the mass of that part of the lever from the knife edge to the end distant a feet away;
m_2, the mass of that part of the lever from the knife edge to the end distant b feet away;
$\overline{X}$, the distance in feet from the knife edge to the center of gravity of the entire lever;
$\bar{x}_1$, the distance in feet from the knife edge to the center of gravity of the mass m_1; and
x_2, the distance in feet from the knife edge to the center of gravity of the mass m_2.
Then, using the foot as our unit of length, we have

$$M\overline{X} = m_2\bar{x} - m_1\bar{x}_1;$$

or

$$\left[\frac{1}{2}\left(\frac{m+n}{12}\right)a\frac{t}{12}d + \frac{1}{2}\left(\frac{m+n}{12}\right)b\frac{t}{12}d\right]\overline{X}$$

$$= \left[\frac{1}{2}\left(\frac{m+n}{12}\right)b\frac{t}{12}d\right]\left[\frac{b}{3}\frac{\left(\frac{m+2n}{12}\right)}{\left(\frac{m+n}{12}\right)}\right] - \left[\frac{1}{2}\left(\frac{m+n}{12}\right)a\frac{t}{12}d\right]\left[\frac{a}{3}\frac{\left(\frac{m+2n}{12}\right)}{\left(\frac{m+n}{12}\right)}\right].$$

Hence,

$$\overline{X} = \frac{1}{3}\,\frac{(m+2n)}{(m+n)}\,(b-a),$$

which is to be measured in the direction of whichever is the greater, b or a.

NUMBER THEORY.

240. Proposed by J. W. NICHOLSON, Louisiana State University.

If the roots of $x^3 - px + q = 0$ are rational, prove that $4p - 3x^2$ is a perfect square.

SOLUTION BY WILLIAM E. PATTEN, Government Institute of Technology, Shanghai, China.

Let the roots of $x^3 - px + q = 0$ be x_1, x_2, x_3. Then

$$x_1 + x_2 + x_3 = 0;$$

and

$$x_1x_2 + x_2x_3 + x_3x_1 = -p.$$

Therefore,

$$\begin{aligned} 4p - 3x_1^2 &= -4(x_1x_2 + x_2x_3 + x_3x_1) - 3x_1^2 \\ &= -4x_2x_3 - 4x_1(x_2 + x_3) - 3x_1^2 \\ &= -4x_2x_3 - 4(-x_2 - x_3)(x_2 + x_3) - 3(-x_2 - x_3)^2 \\ &= -4x_2x_3 + (x_2 + x_3)^2 = (x_2 - x_3)^2. \end{aligned}$$

Since the coefficient of the leading term of the given equation is unity, and the roots are rational, they are also integral.

Therefore, $x_2 - x_3$ is an integer, and $4p - 3x_1^2$ is a perfect square.

Similarly for the other roots:

$$4p - 3x_2^2 = (x_3 - x_1)^2, \qquad \text{and} \qquad 4p - 3x_3^2 = (x_1 - x_2)^2.$$

Also solved by H. N. Carleton, J. E. Rowe, L. C. Mathewson, A. H. Holmes, Elijah Swift, O. S. Adams, J. Rosenbaum, Louis Clark, E. E. Whiteford, and H. S. Uhler.

241. Proposed by CLIFFORD N. MILLS, Brookings, S. Dak.

If $a^2 + b^2 = c^2$, where a, b, and c are integers, then prove that abc will be a multiple of 60.

Solution by Albert G. Caris, Defiance College.

From the well-known theorem that any integral solution of $a^2 + b^2 = c^2$ may be put in the form $2xy$, $x^2 - y^2$, $x^2 + y^2$, where x and y are integers, it follows immediately that

$$abc = 2xy(x - y)(x + y)(x^2 + y^2).$$

Showing that this product is always a multiple of 3, 4, and 5 is sufficient to prove the proposed problem.

I. We may write $x = 2m - 1$, or $2m$ and $y = 2n - 1$, or $2n$, where m and n are integers. Whenever $x = 2m$, or $y = 2n$, abc is a multiple of 4. In all other cases $x = 2m - 1$ at the same time that $y = 2n - 1$ and consequently, $x - y$, $x + y$, and $x^2 + y^2$ are all multiples of 2. Therefore abc is always a multiple of 4.

II. We may write $x = 3r - 1$, $3r$, or $3r + 1$ and $y = 3s - 1$, $3s$, or $3s + 1$, where r and s are integers. Whenever $x = 3r$, or $y = 3s$, abc is a multiple of 3. The combinations resulting from all other cases may be arranged in the two groups below:

Group A		*Group B*	
x	y	x	y
$3r - 1$	$3s - 1$	$3r - 1$	$3s + 1$
$3r + 1$	$3s + 1$	$3r + 1$	$3s - 1$

From combinations of group A the factor $x - y = 3(r - s)$. From combinations of group B the factor $x + y = 3(r + s)$. Therefore abc is always a multiple of 3.

III. We may write

$$x = 5u - 2,\quad 5u - 1,\quad 5u,\quad 5u + 1,\quad \text{or}\quad 5u + 2$$

and

$$y = 5v - 2,\quad 5v - 1,\quad 5v,\quad 5v + 1,\quad \text{or}\quad 5v + 2,$$

where u and v are integers. Whenever $x = 5u$, or $y = 5v$, abc is a multiple of 5. The combinations resulting from all other cases may be arranged in the three groups below:

Group C		*Group D*		*Group E*	
x	y	x	y	x	y
$5u - 2$	$5v - 2$	$5u - 2$	$5v + 2$	$5u \pm 2$	$5v \pm 1$
$5u - 1$	$5v - 1$	$5u - 1$	$5v + 1$		
$5u + 1$	$5v + 1$	$5u + 1$	$5v - 1$	$5u \pm 1$	$5v \pm 2$
$5u + 2$	$5v + 2$	$5u + 2$	$5v + 2$		

All combinations of group C make $x - y = 5(u - v)$. All combinations of group D make $x + y = 5(u + v)$. All combinations of group E make $x^2 + y^2 = 5(5u^2 \pm 4u + 5v^2 \pm 2v + 1)$ or $5(5u^2 \pm 2u + 5v^2 \pm 4v + 1)$. Therefore, abc is always a multiple of 5. Hence, abc is always divisible by 60.

Also solved by S. A. Corey, A. H. Holmes, Horace Olson, J. W. Clawson, J. Rosenbaum, J. E. Rowe, H. C. Feemster, H. N. Carleton, W. J. Thome, Elijah Swift, and E. E. Whiteford.

243. Proposed by CLIFFORD N. MILLS, Brookings, South Dakota.

Determine rational values of x that will render $x^3 + px^2 + qx + r$ a perfect cube. Apply the result to $x^3 - 8x^2 + 12x - 6$.